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Note

On the Degree of Approximation by Linear Positive Operators

1. In a recent paper [4], Shisha and Mond, in giving a quantitative formulation for a well-known result of Korovkin [1], showed how the rate of convergence of linear positive operators to a given continuous function on a closed and bounded interval can be estimated from the rates of convergence to the three simple functions 1, x, and x^2. The purpose of this note is to point out how the method of proof of [4] is easily modified to yield a more general and often better result.

2. For simplicity we utilize the notation of [4].

THEOREM. *Let A be a positive number. Let $-\infty < a < b < \infty$, and let $L_1, L_2, \ldots$ be linear positive operators, all having the same domain D which contains the restrictions of 1, t, t^2 to $[a, b]$. For $n = 1, 2, \ldots$, suppose $L_n(1)$ is bounded. Let $f \in D$ be continuous in $[a, b]$, with modulus of continuity ω. Then for $n = 1, 2, \ldots$,*

$$\|f - L_n(f)\| \leqslant \|f\| \cdot \| L_n(1) - 1 \| + \| L_n(1) + A^{-2} \| \, \omega(A\mu_n), \tag{1}$$

where

$$\mu_n = \| L_n([t - x]^2)(x)\|^{1/2} \tag{2}$$

and $\| \ \|$ stands for the sup norm over $[a, b]$. In particular, if $L_n(1) = 1$, as is often the case, (1) *reduces to*

$$\|f - L_n(f)\| \leqslant (1 + A^{-2}) \, \omega(A\mu_n). \tag{3}$$

Proof. Let $x \in [a, b]$, and let δ be a positive number. Let $t \in [a, b]$. If $|t - x| > \delta$, then $|f(t) - f(x)| \leqslant \omega(|t - x|) = \omega(|t - x| \delta\delta^{-1}) \leqslant (1 + |t - x| \delta^{-1}) \, \omega(\delta) \leqslant [1 + (t - x)^2 \delta^{-2}] \, \omega(\delta)$. The inequality

$$|f(t) - f(x)| \leqslant [1 + (t - x)^2 \delta^{-2}] \, \omega(\delta)$$

holds, obviously, also if $|t - x| \leqslant \delta$. Let n be a positive integer. Then

$$|(L_n(f) - f(x)L_n(1))(x)| \leqslant \omega(\delta)[(L_n(1) + \delta^{-2}L_n([t-x]^2)(x)]$$
$$\leqslant \omega(\delta)[L_n(1)(x) + (\mu_n/\delta)^2].$$

Now take $\delta = A\mu_n$ (by (1) of [4], we can assume $\mu_n > 0$). Then

$$|(L_n(f) - f(x)\, L_n(1))(x)| \leqslant \omega(A\mu_n) \| L_n(1) + A^{-2} \|$$
$$| -f(x) + f(x)\, L_n(1)(x)| \leqslant \|f\| \cdot \| L_n(1) - 1 \| . \qquad (4)$$

Adding, we obtain (1). Note that if we take $A = 1$, the theorem and its proof reduce to those given by Shisha and Mond [4].

EXAMPLE. Let D be the set of all real functions with domain [0, 1]. For $n = 1, 2, \ldots$, let L_n be the linear positive operator with domain D, defined by

$$(L_n\phi)(x) \equiv \sum_{i=0}^{n} \phi(i/n)\binom{n}{i} x^i(1-x)^{n-i}.$$

Let f be a real function with domain [0, 1], continuous there, with modulus of continuity ω. Let n be a positive integer. Then $L_n(1) = 1$, $[L_n(t)](x) \equiv x$, $[L_n(t^2)](x) \equiv (n-1)\, n^{-1}x^2 + n^{-1}x$, $(L_n([t-x]^2))(x) \equiv n^{-1}(x - x^2)$. Taking $A = 2$, our theorem gives

$$\max_{0 \leqslant x \leqslant 1} |f(x) - (L_nf)(x)| \leqslant (1 + 1/4)\, \omega(2/(2n^{1/2})) = (5/4)\, \omega(n^{-1/2}). \qquad (5)$$

Thus, by selecting $A = 2$, our theorem yields the bound for the rate of convergence of Bernstein polynomials given in [2, 3]; whereas in [4], since A is always taken as 1, as good a result is not achieved.

3. In [5], it is shown how a bound for the rate of convergence of linear positive operators to continuous periodic functions can be obtained from the rates of convergence to 1, cos x, and sin x. In an analogous manner, the result there can also be generalized and often improved by introducing an appropriate constant A in the inequalities corresponding to (1) and (3).

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